3-4 Linear Programming

Main Ideas

- Find the maximum and minimum values of a function over a region.
- Solve real-world problems using linear programming.

New Vocabulary

constraints
feasible region
bounded
vertex
unbounded
linear programming

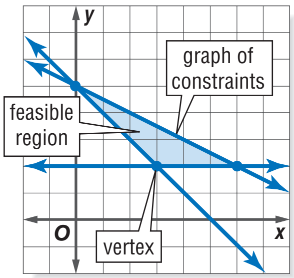
GET READY for the Lesson

The U.S. Coast Guard maintains the buoys that ships use to navigate. The ships that service buoys are called *buoy tenders.*

Suppose a buoy tender can carry up to 8 replacement buoys. The crew can repair a buoy in 1 hour and replace a buoy in $2\frac{1}{2}$ hours.

Maximum and Minimum Values The buoy tender captain can use a system of inequalities to represent the limits of time and replacements on the ship. If these inequalities are graphed, the points in the intersection are combinations of repairs and replacements that can be scheduled.

The inequalities are called the **constraints**. The intersection of the graphs is called the **feasible region**. When the graph of a system of constraints is a polygonal region like the one graphed at the right, we say that the region is **bounded**.



Since the buoy tender captain wants to service the maximum number of buoys, he will need to find the maximum value of the function for points in the feasible region. The maximum or minimum value of a related function *always* occurs at a **vertex** of the feasible region.

EXAMPLE 1 Bounded Region

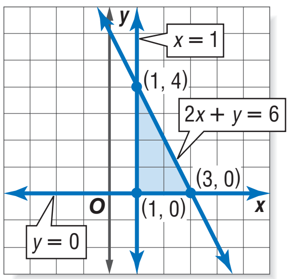
Graph the following system of inequalities. Name the coordinates of the vertices of the feasible region. Find the maximum and minimum values of the function for this region.

$x \geq 1$

$y \geq 0$

$2x + y \leq 6$

$f(x, y) = 3x + y$



Step 1 Graph the inequalities. The polygon formed is a triangle with vertices at (1, 4), (3, 0), and (1, 0).

Reading Math

Function Notation
The notation $f(x, y)$ is used to represent a function with two variables x and y. It is read *f of x and y*.

AFP/CORBIS

Step 2 Use a table to find the maximum and minimum values of $f(x, y)$.

Substitute the coordinates of the vertices into the function.

(x, y)	$3x + y$	$f(x, y)$
(1, 4)	3(1) + 4	7
(3, 0)	3(3) + 0	9
(1, 0)	3(1) + 0	3

← maximum

← minimum

The maximum value is 9 at (3, 0). The minimum value is 3 at (1, 0).

Study Tip

Common Misconception

Do not assume that there is no minimum value if the feasible region is unbounded below the line, or that there is no maximum value if the feasible region is unbounded above the line.

CHECK Your Progress

1. $x \leq 2$

$3x - y \geq -2$

$y \geq x - 2$

$f(x, y) = 2x - 3y$

Sometimes a system of inequalities forms a region that is open. In this case, the region is said to be **unbounded**.

EXAMPLE 2 Unbounded Region

Graph the following system of inequalities. Name the coordinates of the vertices of the feasible region. Find the maximum and minimum values of the function for this region.

$2x + y \geq 3$

$3y - x \leq 9$

$2x + y \leq 10$

$f(x, y) = 5x + 4y$

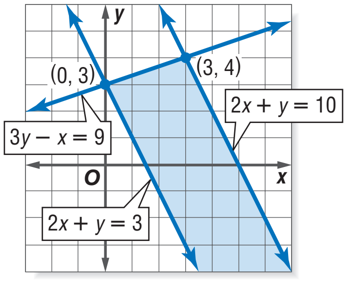
Review Vocabulary

Feasible

Everyday Use possible or likely

Math Use the area of a graph where it is possible to find a solution to a system of inequalities

Graph the system of inequalities. There are only two points of intersection, (0, 3) and (3, 4).



(x, y)	$5x + 4y$	$f(x, y)$
(0, 3)	5(0) + 4(3)	12
(3, 4)	5(3) + 4(4)	31

The maximum is 31 at (3, 4).

Although $f(0, 3)$ is 12, it is not the minimum value since there are other points in the solution that produce lesser values. For example, $f(3, -2) = 7$ and $f(20, -35) = -40$. It appears that because the region is unbounded, $f(x, y)$ has no minimum value.

CHECK Your Progress

2. $g \leq -3h + 4$

$g \geq -3h - 6$

$g \geq \frac{1}{3}h - 6$

$f(g, h) = 2g - 3h$

Linear Programming The process of finding maximum or minimum values of a function for a region defined by inequalities is called **linear programming**.

KEY CONCEPT — *Linear Programming Procedure*

Step 1 Define the variables.

Step 2 Write a system of inequalities.

Step 3 Graph the system of inequalities.

Step 4 Find the coordinates of the vertices of the feasible region.

Step 5 Write a linear function to be maximized or minimized.

Step 6 Substitute the coordinates of the vertices into the function.

Step 7 Select the greatest or least result. Answer the problem.

Real-World Link

Animal surgeries are usually performed in the morning so that the animal can recover throughout the day while there is plenty of staff to monitor its progress.

Source: www.vetmedicine.miningco.com

Concepts in Motion

Animation algebra2.com

Real-World EXAMPLE — Linear Programming

3 **VETERINARY MEDICINE** **As a receptionist for a veterinarian, one of Dolores Alvarez's tasks is to schedule appointments. She allots 20 minutes for a routine office visit and 40 minutes for a surgery. The veterinarian cannot do more than 6 surgeries per day. The office has 7 hours available for appointments. If an office visit costs \$55 and most surgeries cost \$125, how can she maximize the income for the day?**

Step 1 Define the variables.

v = the number of office visits
s = the number of surgeries

Step 2 Write a system of inequalities.

Since the number of appointments cannot be negative, v and s must be nonnegative numbers.

$v \geq 0$ and $s \geq 0$

An office visit is 20 minutes, and a surgery is 40 minutes. There are 7 hours available for appointments.

$20v + 40s \leq 420$ 7 hours = 420 minutes

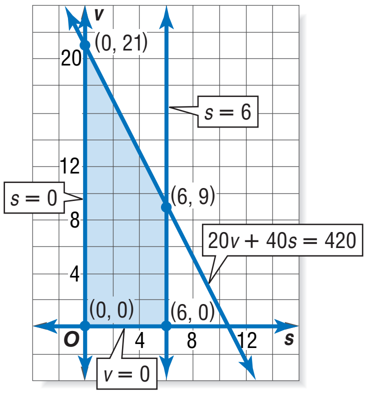
The veterinarian cannot do more than 6 surgeries per day.

$s \leq 6$

Step 3 Graph the system of inequalities.

Step 4 Find the coordinates of the vertices of the feasible region.

From the graph, the vertices of the feasible region are at (0, 0), (6, 0), (6, 9), and (0, 21). If the vertices could not be read from the graph easily, we could also solve a system of equations using the boundaries of the inequalities.



Study Tip

Reasonableness

Check your solutions for reasonableness by thinking of the situation in context. Surgeries provide more income than office visits. So to maximize income, the veterinarian would do the most possible surgeries in a day.

Step 5 Write a function to be maximized or minimized.

The function that describes the income is $f(s, v) = 125s + 55v$. We wish to find the maximum value for this function.

Step 6 Substitute the coordinates of the vertices into the function.

(s, v)	$125s + 55v$	$f(s, v)$
(0, 0)	125(0) + 55(0)	0
(6, 0)	125(6) + 55(0)	750
(6, 9)	125(6) + 55(9)	1245
(0, 21)	125(0) + 55(21)	1155

Step 7 Select the greatest or least result. Answer the problem.

The maximum value of the function is 1245 at (6, 9). This means that the maximum income is $1245 when Dolores schedules 6 surgeries and 9 office visits.

CHECK Your Progress

3. **BUSINESS** A landscaper balances his daily projects between small landscape jobs and mowing lawns. He allots 30 minutes per lawn and 90 minutes per small landscape job. He works at most ten hours per day. The landscaper earns $35 per lawn and $125 per landscape job. He cannot do more than 3 landscape jobs per day and get all of his mowing done. Find a combination of lawns mowed and completed landscape jobs per week that will maximize income.

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CHECK Your Understanding

Example 1 (pp. 138–139)

Graph each system of inequalities. Name the coordinates of the vertices of the feasible region. Find the maximum and minimum values of the given function for this region.

1. $y \geq 2$
   $x \geq 1$
   $x + 2y \leq 9$
   $f(x, y) = 2x - 3y$

2. $y \leq 2x + 1$
   $1 \leq y \leq 3$
   $x + 2y \leq 12$
   $f(x, y) = 3x + y$

3. $x \leq 5$
   $y \geq -2$
   $y \leq x - 1$
   $f(x, y) = x - 2y$

4. $y \geq -x + 3$
   $1 \leq x \leq 4$
   $y \leq x + 4$
   $f(x, y) = -x + 4y$

Example 2 (p. 139)

5. $y \geq -x + 2$
   $2 \leq x \leq 7$
   $y \leq \frac{1}{2}x + 5$
   $f(x, y) = 8x + 3y$

6. $x + 2y \leq 6$
   $2x - y \leq 7$
   $x \geq -2, y \geq -3$
   $f(x, y) = x - y$

7. $x \geq -3$
   $y \leq 1$
   $3x + y \leq 6$
   $f(x, y) = 5x - 2y$

8. $y \leq x + 2$
   $y \leq 11 - 2x$
   $2x + y \geq -7$
   $f(x, y) = 4x - 3y$

Example 3 (pp. 139–140)

MANUFACTURING For Exercises 9–14, use the following information.
The Future Homemakers Club is making canvas tote bags and leather tote bags for a fund-raiser. They will line both types of tote bags with canvas and use leather for the handles of both. For the canvas bags, they need 4 yards of canvas and 1 yard of leather. For the leather bags, they need 3 yards of leather and 2 yards of canvas. Their advisor purchased 56 yards of leather and 104 yards of canvas.

9. Let c represent the number of canvas bags and let ℓ represent the number of leather bags. Write a system of inequalities for the number of bags that can be made.

10. Draw the graph showing the feasible region.

11. List the coordinates of the vertices of the feasible region.

12. If the club plans to sell the canvas bags at a profit of \$20 each and the leather bags at a profit of \$35 each, write a function for the total profit on the bags.

13. How can the club make the maximum profit?

14. What is the maximum profit?

Exercises

HOMEWORK HELP	
For Exercises	See Examples
15–20	1
21–27	2
28–33	3

Graph each system of inequalities. Name the coordinates of the vertices of the feasible region. Find the maximum and minimum values of the given function for this region.

15. $y \geq 1$
$x \leq 6$
$y \leq 2x + 1$
$f(x, y) = x + y$

16. $y \geq -4$
$x \leq 3$
$y \leq 3x - 4$
$f(x, y) = x - y$

17. $y \geq 2$
$1 \leq x \leq 5$
$y \leq x + 3$
$f(x, y) = 3x - 2y$

18. $y \geq 1$
$2 \leq x \leq 4$
$x - 2y \geq -4$
$f(x, y) = 3y + x$

19. $y \leq x + 6$
$y + 2x \geq 6$
$2 \leq x \leq 6$
$f(x, y) = -x + 3y$

20. $x - 3y \geq -7$
$5x + y \leq 13$
$x + 6y \geq -9$
$3x - 2y \geq -7$
$f(x, y) = x - y$

21. $x + y \geq 4$
$3x - 2y \leq 12$
$x - 4y \geq -16$
$f(x, y) = x - 2y$

22. $y \geq x - 3$
$y \leq 6 - 2x$
$2x + y \geq -3$
$f(x, y) = 3x + 4y$

23. $2x + 3y \geq 6$
$3x - 2y \geq -4$
$5x + y \geq 15$
$f(x, y) = x + 3y$

24. $2x + 2y \geq 4$
$2y \geq 3x - 6$
$4y \leq x + 8$
$f(x, y) = 3y + x$

25. $x \geq 0$
$y \geq 0$
$x + 2y \leq 6$
$2y - x \leq 2$
$x + y \leq 5$
$f(x, y) = 3x - 5y$

26. $x \geq 2$
$y \geq 1$
$x - 2y \geq -4$
$x + y \leq 8$
$2x - y \leq 7$
$f(x, y) = x - 4y$

27. **RESEARCH** Use the Internet or other reference to find an industry that uses linear programming. Describe the restrictions or constraints of the problem and explain how linear programming is used to help solve the problem.

PRODUCTION **For Exercises 28–33, use the following information.**
The total number of workers' hours per day available for production in a skateboard factory is 85 hours. There are 40 workers' hours available for finishing decks and quality control each day. The table shows the number of hours needed in each department for two different types of skateboards.

Skateboard Manufacturing Time

Board Type	Production Time	Deck Finishing/ Quality Control
Pro Boards	$1\frac{1}{2}$ hours	2 hours
Specialty Boards	1 hour	$\frac{1}{2}$ hour

28. Let g represent the number of pro boards and let c represent the number of specialty boards. Write a system of inequalities to represent the situation.

29. Draw the graph showing the feasible region.

30. List the coordinates of the vertices of the feasible region.

31. If the profit on a pro board is \$50 and the profit on a specialty board is \$65, write a function for the total profit on the skateboards.

32. Determine the number of each type of skateboard that needs to be made to have a maximum profit.

33. What is the maximum profit?

FARMING **For Exercises 34–37, use the following information.**
Dean Stadler has 20 days in which to plant corn and soybeans. The corn can be planted at a rate of 250 acres per day and the soybeans at a rate of 200 acres per day. He has 4500 acres available for planting these two crops.

34. Let c represent the number of acres of corn and let s represent the number of acres of soybeans. Write a system of inequalities to represent the possible ways Mr. Stadler can plant the available acres.

35. Draw the graph showing the feasible region and list the coordinates of the vertices of the feasible region.

36. If the profit is \$26 per acre on corn and \$30 per acre on soybeans, how much of each should Mr. Stadler plant? What is the maximum profit?

37. How much of each should Mr. Stadler plant if the profit on corn is \$29 per acre and the profit on soybeans is \$24 per acre? What is the maximum profit?

38. **MANUFACTURING** The Cookie Factory wants to sell chocolate chip and peanut butter cookies in combination packages of 6–12 cookies. At least three of each type of cookie should be in each package. The cost of making a chocolate chip cookie is 19¢, and the selling price is 44¢ each. The cost of making a peanut butter cookie is 13¢, and the selling price is 39¢. How many of each type of cookie should be in each package to maximize the profit?

EXTRA PRACTICE
See pages 896, 928.
Math Online
Self-Check Quiz at algebra2.com

H.O.T. Problems

39. **OPEN ENDED** Create a system of inequalities that forms a bounded region.

40. **REASONING** Determine whether the following statement is *always*, *sometimes*, or *never* true.

A function defined by a feasible region has a minimum and a maximum value.

41. **Which One Doesn't Belong?** Given the following system of inequalities, which ordered pair does not belong? Explain your reasoning.

$$y \leq \frac{1}{2}x + 5 \qquad y < -3x + 7 \qquad y \geq -\frac{1}{3}x - 2$$

(0, 0) (−2, 6) (−3, 2) (1, −1)

42. **CHALLENGE** The vertices of a feasible region are $A(1, 2)$, $B(5, 2)$, and $C(1, 4)$. Write a function where A is the maximum and B is the minimum.

43. ***Writing in Math*** Use the information about buoy tenders on page 138 to explain how linear programming can be used in scheduling work. Include a system of inequalities that represents the constraints that are used to schedule buoy repair and replacement and an explanation of the linear function that the buoy tender captain would wish to maximize.

STANDARDIZED TEST PRACTICE

44. **ACT/SAT** For a game she's playing, Liz must draw a card from a deck of 26 cards, one with each letter of the alphabet on it, and roll a six-sided die. What is the probability that Liz will roll an odd number and draw a letter in her name?

A $\frac{2}{3}$ **B** $\frac{1}{13}$ **C** $\frac{1}{26}$ **D** $\frac{3}{52}$

45. **REVIEW** Which of the following best describes the graphs of $y = 3x - 5$ and $4y = 12x + 16$?

F The lines have the same y-intercept.

G The lines have the same x-intercept.

H The lines are perpendicular.

J The lines are parallel.

Spiral Review

Solve each system of inequalities by graphing. (Lesson 3-3)

46. $2y + x \geq 4$
$y \geq x - 4$

47. $3x - 2y \leq -6$
$y \leq \frac{3}{2}x - 1$

Solve each system of equations by using either substitution or elimination. (Lesson 3-2)

48. $4x + 5y = 20$
$5x + 4y = 7$

49. $6x + y = 15$
$x - 4y = -10$

50. $3x + 8y = 23$
$x - y = 4$

51. **CARD COLLECTING** Nathan has 50 baseball cards in his collection from the 1950's and 1960's. His goal is to buy 2 more cards each month. Write an equation that represents how many cards Nathan will have in his collection in x months if he meets his goal. (Lesson 2-4)

GET READY for the Next Lesson

PREREQUISITE SKILL **Evaluate each expression if $x = -2$, $y = 6$, and $z = 5$.** (Lesson 1-1)

52. $x + y + z$

53. $2x - y + 3z$

54. $-x + 4y - 2z$

55. $5x + 2y - z$

56. $3x - y + 4z$

57. $-2x - 3y + 2z$